Transverse localization of directed waves in random media

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In this work we consider the propagation of directed waves in random media with a finite correlation scale in the longitudinal direction. The problem is described by a standard parabolic equation of the same type as the nonstationary Schrödinger equation describing the motion of a quantum particle in a dynamically varying random potential. Applying the path integral approach, we study perturbatively the mean intensity distribution of a pointlike source located in a random medium with inhomogeneities stretched along the propagation direction. We show that in this case the intensity is enhanced on the axis and reduced on the edges of the beam, which can be related to the phenomenon of transverse localization. The dependence of the transverse localization length on the geometry of the problem in different propagation regimes is examined. Though the language of classical waves is used, the results are valid for the quantum case as well.
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I. INTRODUCTION

The propagation of classical waves in random media has been the subject of investigation in various areas of physics for several decades [1–3]. Among different propagation regimes there is a special one when the inhomogeneities of the medium are sufficiently weak, smooth, and large scale as compared to the radiation wavelength. In this case the propagation process is localized mainly in the forward direction (the so-called small-angle scattering) and the backscattering can be fully neglected [1,2]. The propagation of such directed waves can be described to a good approximation by the standard parabolic equation for the complex amplitude $u(\mathbf{r},z)$,

$$2ik\partial_z u+\nabla_{\mathbf{r}}^2 u+k^2\tilde{\varepsilon}(\mathbf{r},z)u(\mathbf{r},z)=0, \tag{1.1}$$

where z is the range coordinate along the main propagation direction and $\mathbf{r}$ is the two-dimensional vector in the transverse plane (cross-range coordinate). The conditions of applicability of this equation in the case of sufficiently strong disorder can be satisfied for anisotropic inhomogeneities when the scattering potential varies very slowly in the z direction. The typical examples are the irregularities in the ionosphere usually stretched along the magnetic field lines or horizontally elongated internal waves in the ocean. In addition, highly anisotropic structures, such as fiber-reinforced composites, have become very important in modern technologies.

The parabolic equation (1.1) coincides with the nonstationary Schrödinger equation

$$i\hbar\partial_t\psi+(\hbar^2/2m)\nabla^2\psi-V(\mathbf{r},t)\psi(\mathbf{r},t)=0, \tag{1.2}$$

which describes the motion of a quantum particle in a random time-dependent potential $V(\mathbf{r},t)$. The analog of time for classical waves is the range coordinate and the random potential corresponds to the spatial fluctuations of the refractive index. This correspondence is defined by the substitutions

$$z\leftrightarrow t,\quad k\leftrightarrow m/\hbar,\quad \tilde{\varepsilon}(\mathbf{r},\tau)\leftrightarrow(-2/m)V(\mathbf{r},t). \tag{1.3}$$

A special property of the parabolic equation, in either classical or quantum wave form, is the unitarity, i.e., the norm $\int d\mathbf{r}|\psi(\mathbf{r},t)|^2$ is prescribed at all times.

It should be mentioned that this problem can be considered in a wider framework including other formulations, which are similar in form. In particular, the imaginary-time version of Eq. (1.2) describes the problem of directed polymers in a random medium [4]. When, in addition, the potential $V(\mathbf{r},t)$ is also imaginary, the model is relevant to quantum tunneling of a strongly localized electron under a random barrier [5].

The original real time model, in both classical and quantum mechanical formulations, was used mainly to study the scaling behavior of the wave motion at large t. If the potential varies very quickly in time, then the problem of finding the statistical characteristics of the wave amplitude $\psi(\mathbf{r},t)$ can be solved by applying the δ-correlation (Markov) approximation [2]. This method has been employed to study the wandering of laser beams in a turbulent atmosphere [6] and diffusion of a quantum particle in dynamically disordered systems for both lattice [7,8] and continuum [9] models. In recent works [10–14] the main efforts were focused on the numerical investigation of the spatial and kinetic characteristics of a quantum particle in a rapidly varying random potential.

Another limiting case is the time-independent potential. In this case the constructive interference of multiple-scattering waves leads to the wave localization phenomenon [3], when the wave is trapped within a finite region of space as in a random resonator. Since the parabolic equation, describing

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the propagation of classical waves, coincides with the Schrödinger equation in two spatial dimensions, we may expect the transition to strong localization for any degree of disorder. This has been demonstrated by using direct numerical calculations in the work of De Raedt, Lagendijk, and de Vries [15], where this effect was called transverse localization. It has been shown that the wave beam propagating in the z direction and having some initial width expands until the beam diameter approaches the transverse localization length. From then on, the beam does not spread in the transverse direction beyond this localization length and propagates without further expansion as in a random waveguide.

A qualitative analysis of the motion of a particle in the intermediate case of finite correlation time has been performed by Bouchaud [16]. It was found that for a correlation time larger than the time needed for the particle to achieve a localization radius, the diffusion of the particle is defined entirely by the time evolution of the scattering potential and by the localization radius in a time-independent potential. Obviously, for zero correlation time the localization is absent because the particle has no time to achieve some static location.

An attempt to analytically obtain the corrections to the Markov approximation has been made by Klyatskin and Tatarskii [17] for the mean field and second-order coherence function and later by Zavorotnyi [18] for the higher statistical moments of the field. In particular, in the latter work a path integral representation for the field was used and the Markov approximation served as a leading term of a perturbative expansion. Evaluation of the next term allowed the applicability limits of the Markov approximation to be considered, but, as in [17], only for the incident plane wave. However, this model leads to the translational invariance of the results in the transverse plane and, consequently, cannot demonstrate any localization behavior. Some related results can be found also in the review paper by Dashen [19]. The mean field (averaged one-particle Green's function) of a directed wave propagating in a medium with a time-independent potential was studied in [20].

In order to consider the transverse localization of directed waves in a random medium with finite correlation in the propagation direction (finite time correlation), we perform an asymptotic analysis of the mean intensity distribution of a pointlike source. The procedure used is similar to that developed in [19] and has been employed in our recent papers [21,22], devoted to the analysis of propagation and localization of classical waves in multiple-scattering random media, without paraxial restriction. As a leading term of the corresponding expansion, we use the straight-line approximation to the path integral solution and estimate the correction perturbatively. Evaluating the second statistical moment of the field (average two-particle Green's function), we have found that the normalized mean intensity in non-Markovian media differs from unity: The mean intensity is enhanced on the axis of the wave beam and is reduced on its edges, which, obviously, can be treated as a direct manifestation of the localization phenomenon.

The outline of this work is as follows. First, in Sec. II, we introduce the parabolic equation and present its solution in a path integral form. The phenomenon of stochastic localization is related to the behavior of the second-order coherence function that is defined in Sec. III. Representing the unknown solution as a perturbative sum of a leading term plus a correction, we obtain an asymptotic expression for the normalized coherence function. The calculation procedure is described in Appendix. Further, in Sec. IV we analyze the wave correction to the mean intensity of a point source. To exemplify the results we evaluate the correction for a Gaussian correlation function. Finally, Sec. V contains a summary and some principal concluding remarks.

II. PATH INTEGRAL FORMULATION

We start with the Helmholtz equation

$$\nabla^2 G + k^2[1+\tilde{\varepsilon}(\mathbf{R})]G(\mathbf{R}|\mathbf{R}_0) = -\delta(\mathbf{R}-\mathbf{R}_0), \tag{2.1}$$

which describes the propagation of a scalar time-harmonic wave in a spatially inhomogeneous medium. The vector $\mathbf{R}$ denotes the position, k is the wave number in a homogeneous medium, and $\varepsilon(\mathbf{R}) = 1+\tilde{\varepsilon}(\mathbf{R})$ is the permittivity distribution, in which $\tilde{\varepsilon}(\mathbf{R})$ is the random perturbation. Assuming that the propagation process takes place mainly in the forward direction, we denote the reduced wave function $g(\mathbf{r},z|\mathbf{r}_0,z_0)$ by extracting the main phase term

$$G(\mathbf{R}|\mathbf{R}_0) = \exp[ik(z-z_0)]g(\mathbf{r},z|\mathbf{r}_0,z_0). \tag{2.2}$$

Neglecting the second range derivative, we transfer from Eq. (2.1) to the standard parabolic equation

$$2ik\partial_z g + \nabla_{\mathbf{r}}^2 g + k^2\tilde{\varepsilon}(\mathbf{r},z)g(\mathbf{r},z|\mathbf{r}_0,z_0) = 0, \tag{2.3a}$$

with the initial condition

$$g(\mathbf{r},z_0|\mathbf{r}_0,z_0) = \delta(\mathbf{r}-\mathbf{r}_0). \tag{2.3b}$$

It is worth noting that this equation is also valid for electromagnetic waves because the polarization does not change essentially in the process of small-angle scattering.

Using the analogy with the Schrödinger equation, we present the solution of Eq. (2.3) in a Feynman path integral form [23]

$$g(\mathbf{r},z|\mathbf{r}_0,z_0) = \int_{\mathbf{r}(z_0)=\mathbf{r}_0}^{\mathbf{r}(z)=\mathbf{r}} D\mathbf{r}(\zeta) \times \exp\left[i\,\frac{k}{2} \int_{z_0}^{z} d\zeta \{[\dot{\mathbf{r}}(\zeta)]^2 + \tilde{\varepsilon}(\mathbf{r}(\zeta),\zeta)\}\right], \tag{2.4}$$

where the integration $\int D\mathbf{r}(\zeta)$ in the continuum of possible trajectories is interpreted as a sum of contributions of arbitrary paths over which the wave propagates from point $\mathbf{r}_0$ in the plane z_0 to point $\mathbf{r}$ in the plane z. Next, we present the virtual trajectory as a sum

$$\mathbf{r}(\zeta) = \overline{\mathbf{r}}(\zeta) + \mathbf{q}(\zeta), \tag{2.5}$$

where the first term

$$\overline{\mathbf{r}}(\zeta) = \frac{z-\zeta}{z-z_0}\,\mathbf{r}_0 + \frac{\zeta-z_0}{z-z_0}\,\mathbf{r} \tag{2.6}$$

is a straight line connecting the end points $\mathbf{r}_0$ and $\mathbf{r}$, and $\mathbf{q}(\zeta)$ is a two-dimensional curved path that is equal to zero at $\zeta = z_0$ and z. As a result, the path integral can be presented as a product of two factors

$$g(\mathbf{r},z|\mathbf{r}_0,z_0)=g_0(\mathbf{r},z|\mathbf{r}_0,z_0)g_\varepsilon(\mathbf{r},z|\mathbf{r}_0,z_0), \tag{2.7}$$

where g_0 is the free-space Green's function

$$g_0(\mathbf{r},z|\mathbf{r}_0,z_0)=\frac{k}{2\pi i(z-z_0)}\exp\left[\frac{ik(\mathbf{r}-\mathbf{r}_0)^2}{2(z-z_0)}\right] \tag{2.8}$$

and the inhomogeneous factor g_ε is given by the expression

$$g_\varepsilon(\mathbf{r},z|\mathbf{r}_0,z_0)=\oint D\mathbf{q}(\zeta) \times\exp\left\{i\,\frac{k}{2}\int_{z_0}^{z} d\zeta\,\tilde{\varepsilon}(\overline{\mathbf{r}}(\zeta)+\mathbf{q}(\zeta),\zeta)\right\}, \tag{2.9}$$

in which the circular integral is used to underline the fact that all the related trajectories are closed in the transverse plane.

III. COHERENCE FUNCTION

The effects related to localization are described by the second-order coherence function (or, in quantum mechanical language, by the two-particle Green's function). We define the normalized coherence function

$$\gamma_2(\mathbf{r}_{\underline{2}},\mathbf{r}_{0\underline{2}};L)=\langle g_\varepsilon(\mathbf{r}_1,z_0+L|\mathbf{r}_{01},z_0)g_\varepsilon^*(\mathbf{r}_2,z_0+L|\mathbf{r}_{02},z_0)\rangle, \tag{3.1}$$

which is equal to unity in a homogeneous medium. Hereafter the angular brackets denote an ensemble average and $\underline{n} = 1,2,...,n$. To simplify the averaging procedure, we assume that the random perturbations $\tilde{\varepsilon}(\mathbf{R})$ are Gaussian. Then the coherence function can be expressed through the correlation (structure) function of the permittivity fluctuations, as is presented in the Appendix, Eqs. (A1)–(A6). Next we extract a straight-line approximation $\overline{\gamma_2}(\mathbf{r}_{\underline{2}},\mathbf{r}_{0\underline{2}};L)$ [Eq. (A8)] and expand the coherence function in a series

$$\gamma_2(\mathbf{r}_{\underline{2}},\mathbf{r}_{0\underline{2}};L)=\overline{\gamma_2}(\mathbf{r}_{\underline{2}},\mathbf{r}_{0\underline{2}};L)\{1+\chi+\cdots\}, \tag{3.2}$$

where the first correction χ is given by

$$\begin{aligned}\chi=\frac{k^2}{2}\int_0^L d\zeta_1\int_0^L d\zeta_2\int d^2s\ F_\varepsilon(\mathbf{s},\zeta_1-\zeta_2)\\ \times\{\tfrac{1}{2}\exp(i\mathbf{v}_1\cdot\mathbf{s})[1-\exp(-i\eta s^2)]\\ +\tfrac{1}{2}\exp(i\mathbf{v}_2\cdot\mathbf{s})[1-\exp(i\eta s^2)]\\ -\exp(i\tilde{\mathbf{v}}\cdot\mathbf{s})[1-\exp(-i\tilde{\eta}s^2)]\}.\end{aligned} \tag{3.3}$$

Here we have denoted the vectors $\mathbf{v}_j$ and $\tilde{\mathbf{v}}$ as

$$\mathbf{v}_j=\overline{\mathbf{r}_j}(\zeta_1)-\overline{\mathbf{r}_j}(\zeta_2),\quad \tilde{\mathbf{v}}=\overline{\mathbf{r}_1}(\zeta_1)-\overline{\mathbf{r}_2}(\zeta_2) \tag{3.4}$$

and we have introduced also a two-dimensional spectral density $F_\varepsilon(\mathbf{s},z)$ of the random structure in the transverse plane; see Eq. (A12). The coefficients η and $\tilde{\eta}$ are given by

$$\eta=\tfrac{1}{2}(L/k)(\zeta/L)(1-\zeta/L) \tag{3.5a}$$

and

$$\tilde{\eta}=\tfrac{1}{2}(L/k)(\zeta/L)(1-2\zeta'/L), \tag{3.5b}$$

where ζ' and ζ are the sum and difference longitudinal coordinates

$$\zeta'=\tfrac{1}{2}(\zeta_1+\zeta_2),\quad \zeta=\zeta_1-\zeta_2. \tag{3.6}$$

Performing spectral expansion with respect to the z coordinate,

$$F_\varepsilon(\mathbf{s},z)=\int d\rho\ \exp(iz\rho)\Phi_\varepsilon(\mathbf{s},\rho), \tag{3.7}$$

we can present the result in terms of the three-dimensional spectrum $\Phi_\varepsilon(\mathbf{s},\rho)$. In δ-correlated (Markovian) media $\Phi_\varepsilon(\mathbf{s},\rho)\equiv\Phi_\varepsilon(\mathbf{s},0)$ and the two spectra used above are related by

$$F_\varepsilon(\mathbf{s},z)=2\pi\delta(z)\Phi_\varepsilon(\mathbf{s},0). \tag{3.8}$$

In this case the correction vanishes and the leading term coincides exactly with the solution of a small-angle approximation of the radiative transfer equation [2]. Therefore, the value of χ describes purely wave nature properties of the propagation process and can be treated as a wave (quantum) correction.

IV. MEAN INTENSITY

We will exemplify the nontrivial properties of the solution (3.2) in non-Markovian media by analyzing the normalized mean intensity $\iota(\mathbf{r},L)$ of a pointlike source in a statistically homogeneous random medium. We can consider the source of the directed beam as a point source when its spatial extent is much smaller than the transverse correlation scale, e.g., the size of the first Fresnel zone $l_F=\sqrt{L/k}$ in the regime of weak intensity fluctuations. In this case the value of $\iota(\mathbf{r},L)$ can be obtained from $\gamma_2(\mathbf{r}_{\underline{2}},\mathbf{r}_{0\underline{2}};L)$ by setting $\mathbf{r}_{0j}=0$ and $\mathbf{r}_j=\mathbf{r}$, which leads to

$$\iota(\mathbf{r},L)=1+\chi+\cdots, \tag{4.1}$$

and the correction χ reduces to the form

$$\begin{aligned}\chi=\frac{k^2}{2}\int_0^L d\zeta_1\int_0^L d\zeta_2\int d^2s\ F_\varepsilon(\mathbf{s},\zeta_1-\zeta_2)\\ \times\cos[\mathbf{r}\cdot\mathbf{s}(\zeta_1-\zeta_2)/L][\cos(\tilde{\eta}s^2)-\cos(\eta s^2)].\end{aligned} \tag{4.2}$$

Performing integration over the sum coordinate ζ' and assuming that the medium is statistically isotropic in the transverse plane, we get

$$\begin{aligned}\chi=k^2\int_0^L d\zeta(L-\zeta)\int_0^\infty ds\ s\ F_\varepsilon(s,\zeta)J_0(rs\zeta/L)\\ \times[(\eta s^2)^{-1}\sin(\eta s^2)-\cos(\eta s^2)],\end{aligned} \tag{4.3}$$

where $J_0(z)$ is the Bessel function. In some practically important situations this formula can be essentially simplified. If all spatial frequencies in the spectrum $F_\varepsilon(s,z)$ satisfy the condition $sl_F \ll 1$ (geometric optics approximation), then we can expand the trigonometric functions in a series and keep only the two first terms, which leads to

$$\chi = \frac{\pi}{6} L^3 \int_0^L d\zeta (\zeta/L)^2 (1-\zeta/L)^3 \times \int_0^\infty ds\ s^5 J_0(rs\zeta/L) F_\varepsilon(s,\zeta). \tag{4.4}$$

In addition, for the longitudinal scale of the medium $l_z \ll L$ we extend the upper limit in the integral over ζ to infinity and approximate the correction by

$$\chi = \frac{\pi}{6} L \int_0^\infty d\zeta\ \zeta^2 \int_0^\infty ds\ s^5 J_0(rs\zeta/L) F_\varepsilon(s,\zeta). \tag{4.5}$$

As an example we will estimate the correction for the anisotropic Gaussian correlation function of the form

$$B_\varepsilon(r,z) = \sigma_\varepsilon^2 \exp(-r^2/l_r^2 - z^2/l_z^2). \tag{4.6}$$

This function corresponds to the spectrum

$$F_\varepsilon(s,z) = (4\pi)^{-1} \sigma_\varepsilon^2 l_r^2 \exp(-l_r^2 s^2/4 - z^2/l_z^2). \tag{4.7}$$

Then the integral in Eq. (4.5) is calculated exactly and we have

$$\chi = (2\sqrt{\pi}/3) \sigma_\varepsilon^2 (l_z^3/l_r^4) L. \tag{4.8}$$

As is natural, the correction increases with the longitudinal scale l_z. However, for isotropic spectrum $l_r = l_z$ and the correction behaves as $\chi \sim \sigma_\varepsilon^2 L/l_z$, i.e., it is greater for smaller inhomogeneities.

Performing integration in Eq. (4.4) for the same Gaussian spectrum leads to

$$\chi = \tfrac{8}{3} \sigma_\varepsilon^2 (L/l_r)^4 \int_0^1 dt\ t^2 (1-t)^3 \times \exp(-\ell^2 t^2)\ {}_1F_1(3,1;-(r/l_r)^2 t^2), \tag{4.9}$$

where $\ell = L/l_z$ is the normalized distance and ${}_1F_1(a,b;z)$ is the hypergeometric function. The results of calculations of the normalized value of χ as a function of the normalized displacement of the observation point from the beam axis r/l_r for several values of ℓ are shown in Fig. 1. In forward direction the correction is positive, and therefore the intensity is enhanced, while it is reduced in other directions. The point where the correction passes through zero can serve as an estimate of the ''transverse localization length.'' This value increases with the decrease of the time correlation l_z, in accordance with the qualitative picture presented in [16].

In the general case, instead of Eq. (4.3) we use an equivalent representation

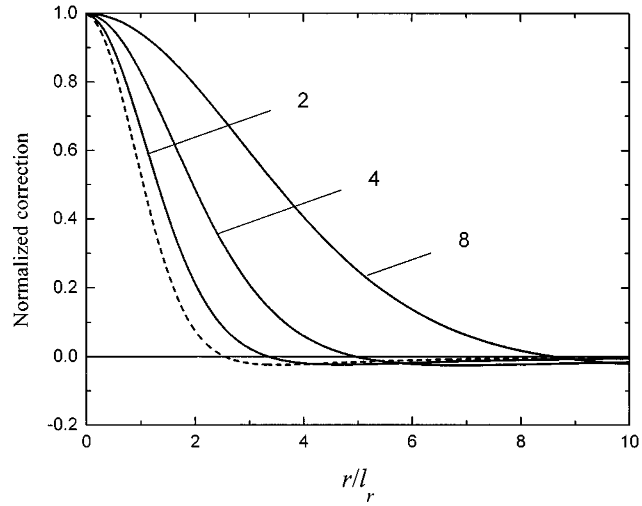

FIG. 1. Normalized wave correction in the geometric optics regime as a function of the normalized transverse coordinate r/l_r for $\ell = 2$, 4, and 8. The dashed line corresponds to the time-independent potential ($\ell = 0$).

$$\chi = k^2 \int_0^L d\zeta (L-\zeta) \int_0^1 d\xi \int_0^\infty ds\ s\ F_\varepsilon(s,\zeta) J_0(rs\zeta/L) \times [\cos(\xi\eta s^2) - \cos(\eta s^2)], \tag{4.10}$$

which seems to be more suitable for numerical calculations. For the Gaussian spectrum (4.7) the correction depends on the dimensionless parameter $\Lambda = l_F/l_r$ and is given by

$$\chi = \sigma_\varepsilon^2 k^2 L^2 \int_0^1 dt \int_0^1 d\xi (1-t) \exp(-\ell^2 t^2) [f(t,\xi) - f(t,1)], \tag{4.11}$$

where

$$f(t,\xi) = (1+a^2)^{-1} \exp(-b)[\cos(ab) + a\ \sin(ab)] \tag{4.12}$$

and

$$a = 2\Lambda^2 t(1-t)\xi, \quad b = (1+a^2)^{-1} (r/l_r)^2 t^2. \tag{4.13}$$

If $\Lambda \ll 1$ we reproduce the results of geometric optics approximation. Moreover, as we can show by direct numerical calculations, this result is also practically exact until $\Lambda = 1$. For $\Lambda \gg 1$ the characteristic scale of the transverse intensity distribution is of the order of l_F. The normalized value of χ for $\ell = 0$ (time-independent potential) as a function of r/l_F for various values of Λ is shown in Fig. 2. The localization length increases with Λ, i.e., it is larger for smaller scales of inhomogeneities in the transverse plane. The same dependence for a fixed value of $\Lambda = 5$ and various values of ℓ is presented in Fig. 3. The variations of the potential in time allow the particle to move a distance of the order of l_r in each time interval l_z and consequently the localization length increases. Finally, in Fig. 4 we present the dependence of χ for a finite value of ℓ normalized to the value of χ for the time-independent potential ($\ell = 0$). We see that for larger Λ the correction is less sensitive to the ''temporal'' variations of the potential.

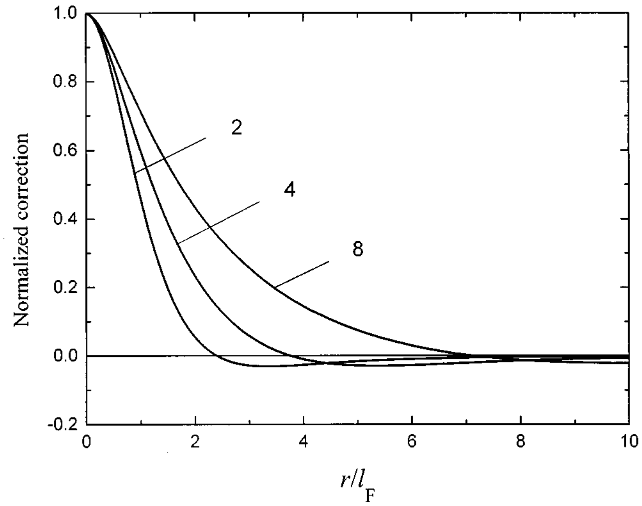

FIG. 2. Normalized wave correction as a function of the normalized transverse coordinate r/l_F for the time-independent potential ($l=0$) and $\Lambda=2$, 4, and 8.

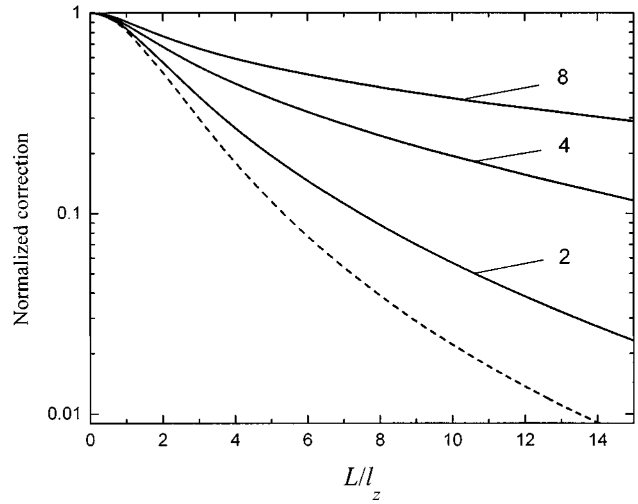

FIG. 4. Normalized wave correction calculated on the beam axis ($r=0$) as a function of ℓ for $\Lambda=2$, 4, and 8. The dashed line corresponds to the geometric optics regime ($\Lambda \ll 1$).

It is worth noting that, in contrast to the visual impression from the data presented in Figs. 1–3, the negative tail, while being very small, is able to compensate for the intensity enhancement in the forward direction. In fact, it is easy to verify that

$$\int d^2r\ \chi(r)=0 \tag{4.14}$$

and the condition of energy conservation (unitarity) is satisfied.

V. SUMMARY

In this work we have implemented the path integral approach for the analysis of transverse localization of directed waves. To this end we have studied the wave correction to the mean intensity of a pointlike source located in a medium with finite correlation along the propagation direction. It has been shown that in such a medium there is a redistribution of the intensity pattern as compared to that in a Markovian medium. In this case the intensity is enhanced on the axis and reduced on the edges of the beam, which can be related to the phenomenon of transverse localization. We have shown, in particular, that the localization length increases with the parameter Λ, i.e., is larger for smaller scales of inhomogeneities in the transverse plane. For larger Λ the correction is also less sensitive to the temporal variations of the potential.

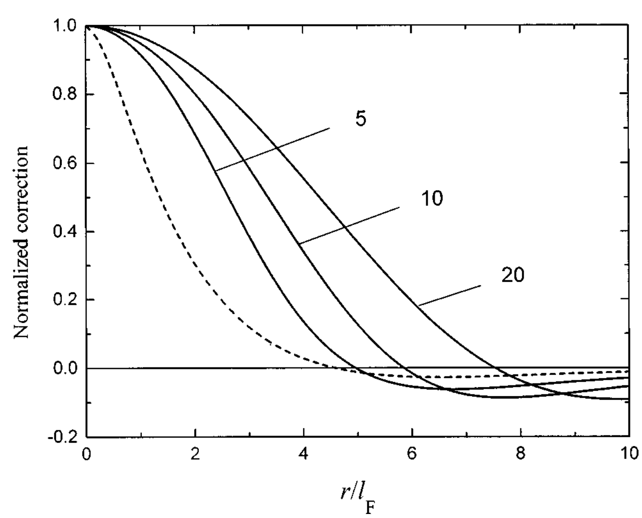

FIG. 3. Normalized wave correction as a function of the normalized transverse coordinate r/l_F for $\Lambda=5$ and $\ell=5$, 10, and 20. The dashed line corresponds to the time-independent potential ($\ell=0$).

On the one hand, the smallness of the correction obtained, i.e., the condition $\chi \ll 1$, can serve as a good test for the applicability of the Markov approximation. For the opposite case $\chi \gg 1$ we may expect a strong transverse localization that has not only a statistical, but also a dynamic nature, i.e., the localization will be observed for almost all realizations of the random medium except for the realizations with measure zero. It is this effect that has been observed in [15] for a single realization of the scattering potential. Such an anisotropic random medium can channel the radiation even in the absence of a deterministic background and can be treated as a random waveguide, a counterpart of a random cavity (or random resonator) for nondirected waves scattered in isotropic media [3,22]. Our results are valid in the intermediate regime, from small χ up to $\chi \approx 1$, and consequently describe the transition to the strong localization behavior. In this case, which can be called a weak transverse localization, the effect is of a stochastic character, but, as always for the localization phenomena, is related to the constructive interference of multiply scattered waves.

The mechanism of transverse localization, whether strong or weak, is universal and can play an important role in wave propagation in many natural and artificial media. As examples, we may note the propagation of electromagnetic HF waves in the ionosphere or UHF waves in the tropospheric layer. The results of recent experiments by Erukhimov *et al.* [24], dealing with oblique chirp sounding of the ionosphere, have revealed the guidance effect of the Pedersen mode in disturbed ionosphere. According to the experimental data, apart from the regular Pedersen mode, a stable ducting mode at lower frequencies has been observed and the effect has been related to anisotropic irregularities in the vicinity of the F-layer maximum. Some indirect indications of the existence of random waveguides in the tropospheric layer are discussed in [25]. In particular, there exists a frequently observed correlation between the magnitude of the signal far beyond the horizon and the variance of turbulent fluctuations of the refractive index in the ground layer. The signal intensity increases with turbulent fluctuations, rather than decreasing as for the regular refractive waveguide. The effect of transverse localization can also play a significant role in sound propagation in the ocean, where the fluctuations of the refractive index are caused mainly by the internal waves that are usually stretched in the horizontal direction. However, for the correct comparison of the theoretical results with the experimental data we have to account for the fractal and anisotropic character of the random structure in the transverse plane and we intend to present the results of appropriate calculations elsewhere.

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APPENDIX

Using the representation (2.9) and introducing the sum and difference vectors

$$\mathbf{p}(\zeta)=\tfrac{1}{2}[\mathbf{q}_1(\zeta)+\mathbf{q}_2(\zeta)],\quad \mathbf{q}(\zeta)=\mathbf{q}_1(\zeta)-\mathbf{q}_2(\zeta),$$

we arrive at the expression for the normalized second-order coherence function

$$\gamma_2(\mathbf{r}_2,\mathbf{r}_{02};L)=\oint D\mathbf{p}(\zeta)\oint D\mathbf{q}(\zeta)\times\exp\left[-\frac{k^2}{4}\int_0^L d\zeta_1\int_0^L d\zeta_2 F_2(\zeta_1,\zeta_2;\mathbf{p}(\zeta),\mathbf{q}(\zeta))\right], \tag{A1}$$

where the scattering function $F_2(\zeta_1,\zeta_2;\mathbf{p}(\zeta),\mathbf{q}(\zeta))$ is given by

$$F_2(\zeta_1,\zeta_2;\mathbf{p}(\zeta),\mathbf{q}(\zeta))=D_\varepsilon(\mathbf{r}_1(\zeta_1)-\mathbf{r}_2(\zeta_2),\zeta_1-\zeta_2)-\frac{1}{2}\sum_{j=1}^{2}D_\varepsilon(\mathbf{r}_j(\zeta_1)-\mathbf{r}_j(\zeta_2),\zeta_1-\zeta_2). \tag{A2}$$

In this formula

$$D_\varepsilon(\mathbf{r}_1-\mathbf{r}_2;z_1-z_2)=\langle[\tilde{\varepsilon}(\mathbf{r}_1,z_1)-\tilde{\varepsilon}(\mathbf{r}_2,z_2)]^2\rangle \tag{A3}$$

is the structure function, which is introduced to describe, among others, the fractal media, in particular, turbulent spectra, for which the correlation function

$$B_\varepsilon(\mathbf{r}_1-\mathbf{r}_2;z_1-z_2)=\langle\tilde{\varepsilon}(\mathbf{r}_1,z_1)\tilde{\varepsilon}(\mathbf{r}_2,z_2)\rangle \tag{A4}$$

diverges at zero. For regular statistically homogeneous media the relation between the correlation and structure functions has the form

$$D_\varepsilon(\mathbf{r};z)=2[B_\varepsilon(0;0)-B_\varepsilon(\mathbf{r};z)]. \tag{A5}$$

Finally, the vectors $\mathbf{r}_j(\zeta)$ in Eq. (A2) are given by

$$\mathbf{r}_j(\zeta)=\overline{\mathbf{r}_j}(\zeta)+[\mathbf{p}(\zeta)+(-1)^{j-1}\mathbf{q}(\zeta)/2],\quad j=1,2. \tag{A6}$$

The leading term of the coherence function is defined by setting $\mathbf{p}(\zeta)=0$ and $\mathbf{q}(\zeta)=0$, which corresponds to the straight-line approximation

$$\overline{\gamma_2}(\mathbf{r}_2,\mathbf{r}_{02};L)=\exp\left[-\frac{k^2}{4}\int_0^L d\zeta_1\int_0^L d\zeta_2 F_2(\zeta_1,\zeta_2;0,0)\right]. \tag{A7}$$

In order to calculate the correction we present the coherence function as

$$\gamma_2(\mathbf{r}_2,\mathbf{r}_{02};L)=\overline{\gamma_2}(\mathbf{r}_2,\mathbf{r}_{02};L)\oint D\mathbf{p}(\zeta)\oint D\mathbf{q}(\zeta)\times\exp\left[\frac{k^2}{4}\int_0^L d\zeta_1\int_0^L d\zeta_2\times\tilde{F}_2(\zeta_1,\zeta_2;\mathbf{p}(\zeta),\mathbf{q}(\zeta))\right], \tag{A8}$$

where

$$\tilde{F}_2(\zeta_1,\zeta_2;\mathbf{p}(\zeta),\mathbf{q}(\zeta))=F_2(\zeta_1,\zeta_2;0,0)-F_2(\zeta_1,\zeta_2;\mathbf{p}(\zeta),\mathbf{q}(\zeta)). \tag{A9}$$

Assuming now the smallness of the argument of the exponential in Eq. (A8), we expand the coherence function in a series

$$\gamma_2(\mathbf{r}_2,\mathbf{r}_{02};L)=\overline{\gamma_2}(\mathbf{r}_2,\mathbf{r}_{02};L)\{1+\chi+\cdots\}, \tag{A10}$$

where the first correction χ is given by

$$\chi=\frac{k^2}{4}\int_0^L d\zeta_1\int_0^L d\zeta_2\oint D\mathbf{p}(\zeta)\oint D\mathbf{q}(\zeta)\times\tilde{F}_2(\zeta_1,\zeta_2;\mathbf{p}(\zeta),\mathbf{q}(\zeta)). \tag{A11}$$

To obtain a soluble quadratic Lagrangian in the path integral we introduce the two-dimensional spectral density $F_\varepsilon(\mathbf{s},z)$ of the random structure in the transverse plane

$$D_\varepsilon(\mathbf{r},z)=2\int d^2s[1-\exp(i\mathbf{r}\cdot\mathbf{s})]F_\varepsilon(\mathbf{s},z). \tag{A12}$$

Then, performing integration in Eq. (A11), we obtain for the wave correction χ the final result, Eq. (3.3).

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